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ROBUSTLY ASYMPTOTIC MEASURE EXPANSIVE CLOSED SETS

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Abstract. In this paper, we show that if a transitive set Λ is robustly asymptotic measure expansive then Λ is hyperbolic.

1. Introduction

Let M be a compact smooth manifold and $f : M \to M$ be a diffeomorphism. For any $\delta > 0$ and $x \in M$, we define $\Gamma_\delta(x) = \{y \in M : d(f^i(x), f^i(y)) \le \delta \ \forall i \in \mathbb{Z}\}$, which is called a *$\delta$-ball.* If a diffeomorphism f is expansive then $\Gamma_\delta(x) = \{x\}$, that is $x = y$. Measure theoretic expansive was suggested by Morales and Sirvent [7] firstly, and a general notion was introduced in [1] which is called asymptotic measure expansive.

Let $\mathcal{M}(M)$ be a Borel probability measure on M. For any $\mu \in \mathcal{M}(M)$ and any closed f-invariant set $\Lambda \subset M$ is said to be *asymptotic μ-expansive* (or Λ is *asymptotic measure expansive*) for f if there is $\delta > 0$ (called an asymptotic measure expansive constant) such that $\mu(f^n(\Gamma_\delta(x))) \to 0$ as $n \to \pm\infty$. If $\Lambda = M$ then we say that f is *asymptotic μ-measure expansive* (or f is asymptotic measure expansive).

It is known that $f^n(\Gamma_\delta(x)) = \Gamma_\delta(f^n(x))$ for all $n \in \mathbb{Z}$. It is a general notion of μ-expansivity (see [1]).

A closed f-invariant set $\Lambda \subset M$ is called *hyperbolic* if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist $C > 0$ and $\lambda \in (0,1)$ such that $\|D_x f^n(v)\| \le C\lambda^n \|v\| (v \in E^s_x \setminus \{0\})$ and $\|D_x f^{-n}(v)\| \le C\lambda^n \|v\| (v \in E^u_x \setminus \{0\})$, for all $x \in \Lambda$ and $n \ge 0$.

A point $x \in M$ is called a *periodic point* if there is $n_x > 0$ such that $f^{n_x}(x) = x$ and called a *non-wandering point* if any neighborhood U_x of

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x there is $n \geq 0$ such that $f^n(U_x) \cap U_x \neq \emptyset$. Let $Per(f)$ be the set of all periodic points of f and $\Omega(f)$ be the set of all non-wandering points of f. It is known that $Per(f) \subset \Omega(f)$. A diffeomorphism f is *Axiom A* if $\overline{Per(f)} = \Omega(f)$ is hyperbolic.

A main interesting issue of differentiable dynamical systems is hyperbolicity of a closed invariant set. A closed f-invariant set $\Lambda \subset M$ is called *robustly asymptotic measure expansive* if there are a C^1 neighborhood $\mathcal{U}$ of f and a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ and for any $g \in \mathcal{U}$, $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is asymptotic measure expansive, where Λ_g is the continuation of Λ.

A closed f-invariant set $\Lambda \subset M$ is called a *transitive set* if there is a point $x \in \Lambda$ such that the omega limit set of x, $\omega_f(x)$, is Λ. Using the notion and a transitive set Λ, Lee and Park [6] proved that if Λ is robustly expansive then Λ is hyperbolic for f. Lee [3] proved that if Λ is robustly continuum-wise expansive then Λ is hyperbolic for f and Lee [5] proved that if Λ is robustly weak measure expansive then Λ is hyperbolic. The results are a motivation about this paper. The following is a main result of this paper.

THEOREM 1.1. *Let $\Lambda \subset M$ be a transitive set of f. If Λ is robustly asymptotic measure expansive for f, then Λ is hyperbolic for f.*

2. Proof of Theorem 1.1

LEMMA 2.1. *Let $A \subset M$ be a closed f-invariant set. If a diffeomorphism $f^k : A \to A$ is the identity map, for some $k \in \mathbb{Z} \setminus \{0\}$ then f^k is not asymptotic measure expansive.*

Proof. Suppose that $f^k : A \to A (k \in \mathbb{Z} \setminus \{0\})$ is asymptotic measure expansive. Let m be the Lebesgue measure on A. For a Borel set $C \subset M$, we define $\nu \in \mathcal{M}(M)$ by

$$\nu(C) = \frac{1}{k}\sum_{i=0}^{k-1} m(f^{-i}(C \cap f^i(A)).$$

Let $\delta > 0$ be the number of the asymptotic measure expansive constant. By continuity, there is $\epsilon > 0$ such that if $d(x, y) < \epsilon (x, y \in M)$ then $d(f^i(x), f^i(y)) < \delta$ for $0 \leq i \leq k-1$. For any $x \in M$, we define $\Gamma_\delta(x) = \{y \in M : d(f^k(x), f^k(y)) \leq \delta,\ \forall k \in \mathbb{Z}\}$. Since $f^k : A \to A$ is the identity map, for any $x \in A$, we see that $\Theta_\delta(x) = \{y \in A : d(x, y) < \epsilon\} \subset \Gamma_\delta(x)$. It is clear that $\nu(f^n(\Theta_\delta(x))) = \nu(\Theta_\delta(x)) > 0$ for all $n \in \mathbb{Z}$. Since

$\nu(f^n(\Gamma_\delta(x))) \to 0$ as $n \to \pm\infty$, but $\nu(f^n(\Theta_\delta(x))) = \nu(\Theta_\delta(x)) \not\to 0$ as $n \to \pm\infty$. This is a contradiction. □

Note that a diffeomorphism f is asymptotic measure expansive if and only if f^k is asymptotic maesure expansive, for all $k \in \mathbb{Z} \setminus \{0\}$. For any closed f-invariant set Λ, we say that Λ satisfies a *local star condition*(or f is local star on Λ) if there are a C^1 neighborhood $\mathcal{U}$ of f and a compact neighborhood U of Λ such that every periodic point in Λ_g is hyperbolic, for any $g \in \mathcal{U}$, where $\Lambda_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Denote by $\mathcal{S}(\Lambda)$ the set of all diffeomorphisms f satisfying a local star condition on Λ.

LEMMA 2.2. *Let $\Lambda \subset M$ be a closed set. If Λ is robustly asymptotic measure-expansive then $f \in \mathcal{S}(\Lambda)$.*

Proof. Let $\mathcal{U}$ and U be the definition of robustly asymptotic measure expansivity. Suppose that by contradiction there exists a diffeomorphism $f \notin \mathcal{S}(\Lambda)$. Then we can take a diffeomorphism $g \in \mathcal{U}$ such that g has a periodic point $p \in \Lambda_g$ which is not hyperbolic. Then $D_p g^{\pi(p)}(g^{\pi(p)}(p) = p)$ has an eigenvalue λ with $|\lambda| = 1$. As in the proof of [3], we assume that $g(p) = p$. Using Franks' Lemma ([2]), there are $r > 0$ and $g_1 \in \mathcal{U}$ such that g_1 has a small arc $\mathcal{J}_p \subset U$ such that for some $k \in \mathbb{Z}$, (i) $g_1^k(\mathcal{J}_p) = \mathcal{J}_p$ and (ii) $g_1^k|_{\mathcal{J}_p} : \mathcal{J}_p \to \mathcal{J}_p$ is the identity map. By Lemma 2.1, g_1^k is not asymptotic measure expansive. This is a contradiction. □

LEMMA 2.3. [4] *Let $\Lambda \subset M$ be a transitive. If f is a local star on Λ, then Λ is hyperbolic.*

End of the Proof of Theorem 1.1 Since Λ is robustly asymptotic measure expansive, by Lemma 2.2, f satisfies a local star on Λ. By Lemma 2.3, Λ is hyperbolic. □

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